System of Difference Constraints

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1 Problem

Given some inequality on some variable $(x_i, x_j, ...)$ in form $x_j - x_i \leq w$, we need to determine whether we can assign values to the variables so that all the given inequalities are satisfiable or not. If satisfiable, then output a solution.

2 Solution

- For each variable we create a vertex.
- For each inequality, $x_j - x_i \leq w$, we give a directed edge (v_i, v_j) with cost w.
- Create a source vertex S and give an edge (S, v_i) for all vertices with cost 0. Can be solved without source vertex if we use SPFA.

The SPFA code for determining existence of negative cycle:

```
bool spfa()
{
	queue<int> Q;
	for(int i=1; i<=n; i++)
	{
		Q.push(i); dist[i] = inf; inq[i] = true; cntr[i] = 1;
	}
	dist[1] = 0;
	while(!Q.empty())
	{
		int u = Q.front();
		Q.pop();
		inq[u] = false;

		for(auto it: graph[u])
		{
			int v = it[0], w = it[1];
			if(dist[v] > dist[u] + w)
			{
				dist[v] = dist[u] + w;
				if(!inq[v])
				{
					inq[v] = true;
					cntr[v]++;
					Q.push(v);
					if(cntr[v]>n) return false;
				}
			}
```

```
            }
        }
        return true;
}
```

If the constraint graph contains a negative cycle, then the system of differences is unsatisfiable.

3 Determining a Possible Solution

- If there is no negative cycle in the constraint graph, then there is a solution for the system.
- For each variable x_i, x_i = shortest path distance of v_i from the source vertex in constraint graph.
- Let $x = x_1, x_2, ..., x_n$ be a solution to a system of difference constraints and let d be any constant. Then $x + d = x_1 + d, x_2 + d, ..., x_n + d$ is a solution as well.
- Shortest Path can be calculated from Bellman-Ford algorithm.
- Bellman-Ford maximizes $x_1 + x_2 + ... + x_n$ subject to the constraints $x_j - x_i \leq w_{ij}$ and $x_i \leq 0$
- Bellman-Ford also minimizes $max(x_i) - min(x_i)$